COMMENTS ON "A THEORY OF ORDINARY P-ADIC CURVES"

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(1.) The notation "$\mathcal{E} \to \tau_{X^{\log}/S^{\log}}$" in [$p$Ord], Chapter II, Proposition 2.6, (1), should read "$\mathrm{Ad}(\mathcal{E}) \to \tau_{X^{\log}/S^{\log}}$".

(2.) With regard to the notation "$(X_T^{\log})_{\mathbb{Q}_p}$", "$(X_P^{\log})_{\mathbb{Q}_p}$" in the paragraph immediately preceding Theorem 0.4, we note the following: Let K be a finite extension of $\mathbb{Q}_p$ and $\mathfrak{Y}$ a formally smooth p-adic formal scheme over the ring of integers $\mathcal{O}_K$ of K, i.e., such as a suitable étale localization of X_T or one of the coverings that occurs in the inverse limit used to define X_P. Then $\mathfrak{Y} \times_{\mathbb{Z}_p} \mathbb{Q}_p$ (i.e., "$\mathfrak{Y} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$") is defined as the *ringed space* obtained by tensoring the structure sheaf of $\mathfrak{Y}$ over $\mathcal{O}_K$ with K. Thus, if, for instance, $\mathfrak{Y}$ is the formal scheme obtained as the formal inverse limit of an inverse system of schemes

$$\ldots \hookrightarrow \mathfrak{Y}_n \hookrightarrow \mathfrak{Y}_{n+1} \hookrightarrow \ldots$$

— where n ranges over the positive integers, and each "$\hookrightarrow$" is a nilpotent thickening — and U is an affine open of the *common* underlying topological space of the $\mathfrak{Y}_n$, then the rings of sections of the respective structure sheaves $\mathcal{O}_{\mathfrak{Y}}$, $\mathcal{O}_Y$ of $\mathfrak{Y}$, Y over U are, by definition, given as follows:

$$\mathcal{O}_{\mathfrak{Y}}(U) \stackrel{\mathrm{def}}{=} \varprojlim_n \mathcal{O}_{\mathfrak{Y}_n}(U); \quad \mathcal{O}_Y(U) \stackrel{\mathrm{def}}{=} \mathcal{O}_{\mathfrak{Y}}(U) \otimes_{\mathcal{O}_K} K.$$

Here, we observe that $\mathcal{O}_{\mathfrak{Y}}(U)$ is the *p-adic completion* of a *normal noetherian ring of finite type over* $\mathcal{O}_K$. In particular, we observe that one may consider *finite étale coverings* of Y, i.e., by considering *systems of finite étale algebras* $\mathcal{A}_U$ over the various $\mathcal{O}_Y(U)$ [that is to say, as U is allowed to vary over the affine opens of the $\mathfrak{Y}_n$] equipped with *gluings* over the intersections of the various U that appear. Note, moreover, that by considering the *normalizations* of the $\mathcal{O}_{\mathfrak{Y}}(U)$ in $\mathcal{A}_U$, we conclude [cf. the discussion of the Remark immediately following Theorem 2.6 in Section II of [Falt]] that

(NorFor) any such system $\{\mathcal{A}_U\}_U$ may be obtained as the $W \stackrel{\mathrm{def}}{=} \mathfrak{W} \times_{\mathcal{O}_K} K$ for some *formal scheme* $\mathfrak{W}$ that is *finite* over $\mathcal{Y}$, and that arises as the *formal inverse limit* of an inverse system of schemes

$$\ldots \hookrightarrow \mathfrak{W}_n \hookrightarrow \mathfrak{W}_{n+1} \hookrightarrow \ldots$$

— where n ranges over the positive integers; each "$\hookrightarrow$" is a nilpotent thickening; for each affine open V of the *common* underlying topological space of the $\mathfrak{W}_n$, $\mathcal{O}_{\mathfrak{W}}(V)$ is the *p-adic completion* of a *normal noetherian ring of finite type over* $\mathcal{O}_K$.

Indeed, this follows from well-known considerations in commutative algebra, which we review as follows. Let R be a *normal noetherian ring of finite type over a complete discrete valuation ring* A [i.e., such as $\mathcal{O}_K$ in the above discussion] with *maximal ideal* $\mathfrak{m}_A$ and *quotient field* F such that R is *separated* in the $\mathfrak{m}_A$-adic topology. Thus, since A is *excellent* [cf. [EGA IV$_2$], Scholie 7.8.3, (iii)], it follows [cf. [EGA IV$_2$], Scholie 7.8.3, (ii)] that R is *excellent*, hence that the $\mathfrak{m}_A$-adic completion $\widehat{R}$ of R is also *normal* [cf. [EGA IV$_2$], Scholie 7.8.3, (v)]. Then it is well-known and easily verified [by applying a well-known argument involving the *trace map*] that the *normalization* of $\widehat{R}$ in any *finite étale algebra* over $\widehat{R} \otimes_A F$ is a *finite algebra* over $\widehat{R}$. Let $\widehat{S}$ be such a *finite algebra* over $\widehat{R}$. Then it follows immediately from a suitable version of *"Hensel's Lemma"* [cf., e.g., the argument of [AbsTpII], Lemma 2.1] that $\widehat{S}$ may be obtained, as the notation suggests, as the $\mathfrak{m}_A$-adic completion of a *finite algebra* S over R, which may in fact be assumed to be *separated* in the $\mathfrak{m}_A$-adic topology and [by replacing S by its normalization and applying [EGA IV$_2$], Scholie 7.8.3, (v), (vi)] *normal*. Let $f \in R$ be an element that maps to a *non-nilpotent* element of $R/\mathfrak{m}_A \cdot R$. Write $R_f \stackrel{\text{def}}{=} R[f^{-1}]$; $S_f \stackrel{\text{def}}{=} S \otimes_R R_f$; $\widehat{R}_f$, $\widehat{S}_f$ for the respective $\mathfrak{m}_A$-adic completions of R_f, S_f. Then it follows again from [EGA IV$_2$], Scholie 7.8.3, (v), that $\widehat{S}_f$, which may be naturally identified [since S is a *finite algebra* over R] with $\widehat{S} \otimes_{\widehat{R}} \widehat{R}_f$, is *normal*. That is to say, it follows immediately that

(NorForZar) the operation of forming *normalizations* [i.e., as in the above discussion] is *compatible* with *Zariski localization* on the *given formal scheme*.

On the other hand, one verifies immediately that (NorFor) follows formally from (NorForZar).

Bibliography

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